Capacity Characterization of UAV-Enabled Multiple Access Channel via Trajectory Optimization

Peiming Li and Jie Xu
School of Information Engineering, Guangdong University of Technology, Guangzhou 510006, China
E-mail: peiminglee@outlook.com, jiexu@gdut.edu.cn

Abstract—This paper considers an unmanned aerial vehicle (UAV)-enabled multiple access channel (MAC), where multiple ground users transmit individual information to a mobile UAV. For the purpose of initial investigation, we consider the linear topology case when the users are deployed in a straight line on the ground and the UAV flies at a fixed altitude in the sky. Under this setup, we aim to characterize the fundamental capacity region of the UAV-enabled MAC over a particular communication period, which is defined as the set of all average rate tuples that are simultaneously achievable by all the users, under the users’ individual maximum power and the UAV’s maximum flying speed constraints. Towards this end, we maximize the average sum rate of all users subject to a set of rate profile constraints, by optimizing the UAV’s one-dimensional (1D) trajectory design. However, this problem is highly non-convex and consists of an infinite number of variables over continuous time; therefore, it is difficult to be solved optimally in general. Despite this difficulty, we present the globally optimal solution to this problem, by first showing that any speed-constrained UAV trajectory is equivalent to the combination of a maximum-speed trajectory and a speed-free trajectory, and then employing the Lagrange duality method. It is shown that the optimal trajectory solution follows an interesting successive hover-and-fly (SHF) structure, i.e., the UAV successively hovers above a number of optimized locations, and flies among them unidirectionally at the maximum speed. Numerical results show that the capacity region achieved by the optimal SHF trajectory design significantly outperforms the achievable rate region by other benchmark schemes.

I. INTRODUCTION

Unmanned aerial vehicle (UAV)-enabled wireless communications have attracted a lot of attentions in both academia and industry, in which UAVs are used as aerial communication platforms (e.g., base stations (BSs) and relays) to provide wireless data access for subscribers on the ground [1]–[8]. Different from conventional terrestrial wireless communication networks, UAVs have various advantages. For example, UAVs can be deployed rapidly in a cost-effective manner in e.g. emergency situations and remote areas. Also, UAVs possess strong line-of-sight (LoS) wireless links with ground users, which help to provide reliable communications between them. Furthermore, by exploiting the fully-controllable mobility, UAVs can properly adjust their flying locations over time (a.k.a. trajectories) as new design degrees of freedom for optimizing the communication performance.

In the literature, there have been various works investigating the joint UAV trajectory design and wireless resource allocation to optimize the performance of UAV-enabled wireless communications under different setups (see, e.g., [9]–[17]). For instance, the authors in [9] first considered the trajectory optimization for throughput maximization in the UAV-enabled relaying system. With data buffering employed, the UAV-relay can fly close to the source for information reception, and then move towards the destination for information forwarding. By reducing the corresponding transmission distances, such trajectory design can significantly improve the end-to-end throughput. Furthermore, [10]–[13] studied the UAV-enabled multiuser communication with one single UAV communicating with multiple ground users. By proper trajectory design, the UAV can sequentially visit these users to shorten the transmission distances, thus maximizing different users’ communication performance in a fair manner. These results were then extended to the multi-UAV-enabled wireless networks in [10], [14]. By jointly designing their trajectories, the multiple UAVs can not only shorten the transmission distances with intended users for better link quality, but also enlarge the distances with undesirable users for mitigating the inter-UAV interference. In addition, there have been other works studying the trajectory design in other UAV-enabled wireless applications such as mobile edge computing [15], wireless power transfer [16], and wireless powered communication networks [17].

Despite the recent research progress, there are rarely prior works characterizing the information-theoretical limits of UAV-enabled wireless communications with trajectory optimization. This problem, however, is very important for understanding the fundamental performance upper bound and guiding the practical design for such systems. In particular, consider the UAV-enabled multiuser communication, namely the multiple access channel (MAC) and broadcast channel (BC), in which one UAV communicates with multiple ground users in the uplink and downlink, respectively. By employing capacity-achieving non-orthogonal multiple access (NOMA) transmission strategies (i.e., successive interference cancellation (SIC) for MAC and superposition coding (SC) with SIC for BC)\(^1\), how to jointly optimize the UAV trajectory design and the wireless resource allocation for maximizing the capacity region is a challenging task that has not been well addressed in the literature yet. To our best knowledge, there is only one prior work [19] that investigated the capacity region of a UAV-enabled two-user BC. It was shown that to maximize

- Please refer to [18] for details on recent advancements in practical implementation of NOMA in fifth-generation (5G) cellular networks.

J. Xu is the corresponding author.
the capacity, the optimal UAV trajectory should follow a hover-fly-hover (HFH) structure, i.e., the UAV successively hovers at a pair of initial and final locations above the line segment of the two users each with a certain amount of time and flies unidirectionally between them at the maximum speed, during which SC is generally needed [19]. Due to the uplink-downlink duality [20], it is expected that such an HFH trajectory, jointly with optimized wireless resource allocation, can also maximize the capacity region of the UAV-enabled two-user MAC. However, it is worth noting that the optimality of the HFH trajectory is verified via a very complicated proof technique based on the “monotonicity” of the two-user BC capacity region under any given UAV locations, while this technique cannot be applied to the general case with more than two ground users. Therefore, how to optimize the UAV trajectory, reveal its optimal structure, and accordingly characterize the capacity region for the general UAV-enabled MAC and BC is a difficult problem that is unaddressed. This thus motivates our investigation in this paper.

For the purpose of exposition, we consider a UAV-enabled MAC with a linear topology, where the users are deployed in a straight line on the ground and the UAV flies at a fixed altitude in the sky. Under this setup, we aim to characterize the fundamental capacity region of the UAV-enabled MAC over a particular communication period, which is defined as the set of all average rate tuples that are simultaneously achievable by all the users, under the users’ individual maximum power and the UAV’s maximum flying speed constraints. Towards this end, we maximize the average sum rate of all users subject to a set of rate profile constraints, by optimizing the UAV’s one-dimensional (1D) trajectory design. However, this problem is highly non-convex and consists of an infinite number of variables over continuous time; therefore, it is difficult to be solved optimally in general. To tackle this issue, we first show that any speed-constrained UAV trajectory is equivalent to the combination of a maximum-speed trajectory and a speed-free trajectory, in terms of their achieved rate region. Accordingly, we transform the original speed-constrained UAV trajectory optimization problem into an equivalent speed-free trajectory optimization problem, which can be optimally solved via the Lagrange duality method. It is shown that to maximize the capacity region of the UAV-enabled MAC, the optimal UAV trajectory solution follows an interesting successive hover-and-fly (SHF) trajectory, i.e., the UAV successively hovers above a number of optimized locations, and flies among them unidirectionally at the maximum speed. Numerical results show that the capacity region achieved by the optimal 1D trajectory design significantly outperforms the achievable rate region by other benchmark schemes.

II. SYSTEM MODEL

In this paper, we consider a UAV-enabled MAC, in which $K > 1$ ground users send individual information to a mobile UAV in the sky, over a finite communication period $T \triangleq (0, T]$ with duration $T > 0$. For initial investigation, we focus on the linear topology case when all the ground users are deployed in a straight line with altitude zero, as shown in Fig. 1, where $(w_k, 0, 0)$ denotes the location of each ground user $k \in K \triangleq \{1, ..., K\}$ in a three-dimensional (3D) Cartesian coordinate system. Here, $w_1 \leq \ldots \leq w_K$ is assumed without loss of generality. It is also assumed that the UAV flies above the line segment of the users at a fixed altitude $H > 0$, with the time-varying location being denoted as $(x(t), 0, H)$ at time instant $t \in T$. Let $V_{\text{max}} \geq 0$ denote the maximum UAV speed in meters/second (m/s). We thus have

$$|\dot{x}(t)| \leq V_{\text{max}}, \quad \forall t \in T,$$

where $\dot{x}(t)$ denotes the first-order derivative of $x(t)$ over time.

It is assumed that the transmission from the ground users to the UAV is implemented over a system bandwidth $B$ in Hertz (Hz), where $T_s = 1/B$ denotes the corresponding symbol duration in second (s). Accordingly, the UAV’s location change within each symbol duration is assumed to be negligible as compared to the UAV’s flying altitude $H$, i.e., $V_{\text{max}}T_s \ll H$ [19]. Thus, the wireless channel from each user to the UAV is unchanged within each symbol interval. As the air-to-ground wireless channels are normally dominated by the LoS link, we consider the free-space path loss model from ground users to the UAV, as commonly adopted in the literature (see, e.g., [9], [19] and the references therein). As a result, the channel power gain from each ground user $k \in K$ to the UAV at time instant $t \in T$ is modeled as

$$h_k(x(t)) = \frac{\gamma_0}{(x(t) - w_k)^2 + H^2},$$

where $\gamma_0$ denotes the channel power gain at the reference distance of $d_0 = 1$ m.

At time instant $t \in T$, let $s_k(t)$ denote the information-bearing signal transmitted by user $k \in K$. Accordingly, the received signal at the UAV is expressed as

$$y(t) = \sum_{k \in K} \sqrt{h_k(x(t))}s_k(t) + n(t),$$

where $n(t)$ denotes the additive white Gaussian noise (AWGN) at the UAV receiver with noise power $\sigma^2$. Under given $\{x(t)\}$, the signal model in (3) resembles a conventional fading MAC consisting of $K$ transmitters (ground users) and one receiver (UAV). In order to achieve the capacity region of this channel, the ground users should employ Gaussian signaling by setting $s_k(t)$’s as independent circularly symmetric complex Gaussian (CSCG) random variables with zero mean and variances

2Notice that our design in this paper can be easily extended in the case with other channel models, as will be detailed in the journal version.

![Fig. 1. Illustration of the UAV-enabled MAC.](image-url)
\[\mathbb{E}(|s_k(t)|^2) = P, \forall k \in K,\] where \(\mathbb{E}(\cdot)\) denotes the statistical expectation, and \(P\) denotes the maximum transmit power at each user.\(^3\) To achieve the channel capacity, the UAV adopts SIC to decode the information from the \(K\) users. Let the permutation \(\pi = [\pi(1), \ldots, \pi(K)]\) denote the decoding order at the UAV, which indicates that the UAV receiver first decodes the information \(s_{\pi(K)}(t)\) transmitted by user \(\pi(K)\), then decodes \(s_{\pi(K-1)}(t)\) by cancelling the interference from \(s_{\pi(K)}(t)\), followed by \(s_{\pi(K-2)}(t)\), \(s_{\pi(K-3)}(t)\), and so on, until \(s_{\pi(1)}(t)\). The achievable rate in bits per second per Hertz (bps/Hz) at user \(\pi(k), k \in K\), under a given decoding order \(\pi\) is then given by

\[
r_{\pi(k)} = \frac{1}{T} \int_0^T \log_2 \left( \frac{\sigma^2 + \sum_{i=1}^k P_{h_{\pi(i)}}(x(t))}{\sigma^2 + \sum_{i=1}^{K-1} P_{h_{\pi(i)}}(x(t))} \right) dt. \tag{4}
\]

By properly designing the decoding order and allowing time-sharing among different decoding orders, the region of all achievable average rate tuples \(r = [r_1, \ldots, r_K]\) in bps/Hz for the \(K\) ground users can be expressed as \([21]\)

\[
\mathcal{C}([x(t)]) = \left\{ r \in \mathbb{R}^+_K \left| \sum_{k \in K} r_k \leq \frac{1}{T} \int_0^T \log_2 \left( 1 + \sum_{k \in K} P_{h_k}(x(t)) \right) dt, \forall \mathcal{K} \subseteq K \right. \right\}. \tag{5}
\]

where \(\mathbb{R}^+_K\) denotes the set of all real non-negative vectors with dimension \(K\).

Let \(\mathcal{X}\) denote the feasible set of \([x(t)]\) specified by the UAV’s speed constraints in (1). Then the capacity region of the UAV-enabled MAC is defined as

\[
\mathcal{C}(V_{\text{max}}, T) = \bigcup_{\{x(t)\} \in \mathcal{X}} \mathcal{C}([x(t)]), \tag{6}
\]

which consists of all the achievable average rate tuples for the ground users over the communication period \(T\), subject to the UAV’s maximum speed constraint in (1).

In this paper, we are interested in characterizing the Pareto (or the upper-right) boundary of the capacity region \(\mathcal{C}(V_{\text{max}}, T)\), at which each user cannot increase its achievable average rate unless sacrificing the rates of other users. Specifically, let \(\alpha = [\alpha_1, \ldots, \alpha_K]\) denote a rate-profile vector which specifies the rate allocation among the \(K\) ground users with \(\alpha_k \geq 0, \forall k \in K\), and \(\sum_{k \in K} \alpha_k = 1\). Here, a larger value of \(\alpha_k\) indicates that ground user \(k\) has a higher priority in information transmission to achieve a larger average rate. Then, the characterization of each Pareto-boundary point on the capacity region corresponds to solving the following problem:

\[(P1): \max_{r,\{x(t)\}, R} R \tag{7}\]

\[
\text{s.t. } r_k \geq \alpha_k R, \forall k \in K \tag{8}
\]

where \(R\) denotes the achievable sum average rate of the \(K\) ground users. Problem (P1) is challenging to be solved optimally due to the following reasons. First, the constraint in (8) is non-convex, as the right-hand-side (RHS) terms of the inequalities in (5) are non-concave with respect to \([x(t)]\). Next, problem (P1) involves an infinite number of optimization variables (i.e., \([x(t)\)’s over continuous time \(t\)]. As a result, (P1) is generally a highly non-convex optimization problem, and there is no standard method in the existing literature to solve this problem optimally. For notational convenience, let \([x^{opt}(t)]\) denote the optimal UAV trajectory solution to problem (P1), and \(r^{opt} = [r_1^{opt}, \ldots, r_K^{opt}]\) denote the correspondingly achieved rate tuples by the \(K\) users.

**Remark 2.1:** Notice that when there are only two users with \(K = 2\), problem (P1) is simplified as

\[(P2): \max_{\{x(t)\}, r_1, r_2, R} R \tag{9}\]

\[
\text{s.t. } r_k \geq \alpha_k R, \forall k \in \{1, 2\} \tag{10}
\]

\[
r_1 \leq \log_2 \left( 1 + P_{h_1}(x(t))/\sigma^2 \right) \tag{11}
\]

\[
r_2 \leq \log_2 \left( 1 + P_{h_2}(x(t))/\sigma^2 \right) \tag{12}
\]

\[
r_1 + r_2 \leq \log_2 \left( 1 + P(h_1(x(t)) + h_2(x(t)))/\sigma^2 \right) \tag{13}
\]

| \(\dot{x}(t)\) | \(\leq V_{\text{max}}, \forall t \in T, \tag{14}\

Due to the uplink-downlink duality, it is evident that problem (P2) is a simplified version of the capacity region characterization problem for the UAV-enabled two-user BC in [19] with fixed power allocation. Similarly as in [19], it follows that the optimal UAV trajectory solution to problem (P2) has the so-called HFH structure, i.e., the UAV first hovers at the initial location \(x_1\) for duration \(t_1\), then flies unidirectionally to the final location \(x_F \geq x_1\) at the maximum speed \(V_{\text{max}}\), and finally hovers at \(x_F\) for the remaining duration \(t_F = T - t_1 - (x_F - x_1)/V_{\text{max}}\). In this case, problem (P2) can be optimally solved via a 3D exhaustive search over the initial and final locations \(x_1\) and \(x_F\), and the hovering duration \(t_1\). It is worth noting that the optimality of the HFH trajectory in [19] is proved based on the “monotonicity” of the capacity region under any given UAV locations. However, this technique cannot be extended to solve (P1) in the general UAV-enabled MAC with \(K > 2\) ground users.

### III. Optimal Solution to Problem (P1)

In this section, we propose a new approach to optimally solve problem (P1). Before proceeding, we first present the following lemma, which follows directly from [19, lemma 3].

**Lemma 3.1:** There always exists a uni-directional UAV trajectory that is an optimal solution to problem (P1), i.e.,

\[x(t_1) \leq x(t_2), \forall t_1, t_2 \in T, t_1 \leq t_2. \tag{15}\

Based on Lemma 3.1, we focus on the uni-directional trajectory to (P1) without loss of optimality. Suppose that the initial and final locations of the trajectory are \(x(0) = x_1\) and \(x(T) = x_F\), respectively, which are optimization variables to be decided later. Here, it must follow that \(w_1 \leq x_1 \leq x_F \leq w_K\), such that the UAV always flies within the line segment.
above the \( K \) ground users’ locations to maximize the capacity region. In the following, we solve problem (P1) by first solving the following problem (P1.1) under any given initial and final locations \( x_1 \) and \( x_F \), and then using a 2D exhaustive search over \([w_1, w_K]\) to find the optimal \( x_1 \) and \( x_F \).

(P1.1): \[
\begin{align*}
\max_{\{x(t)\}_t} & \quad R \\
\text{s.t.} & \quad x_1 \leq x(t) \leq x_F, \quad \forall t \in T
\end{align*}
\]

(10)

In the rest of this section, we will focus on solving problem (P1.1) under any given \( x_1 \) and \( x_F \) with \( w_1 \leq x_1 \leq x_F \leq w_K \). In particular, we first reformulate the speed-constrained trajectory optimization problem (P1.1) as an equivalent speed-free trajectory optimization problem, and then employ the Lagrange duality method to obtain the optimal solution.

A. Problem Formulation of (P1.1)

To facilitate the problem reformulation of (P1.1), we first make the following definitions.

- **Speed-constrained trajectory** \( \{x(t)\}_t \): This is the original trajectory of our interest. At any time instant \( t \in (0, T] \), the UAV flying speed \( \dot{x}(t) \) cannot exceed the limit \( V_{\text{max}} \), i.e., \( |\dot{x}(t)| \leq V_{\text{max}}, \forall t \in (0, T] \).
- **Maximum-speed trajectory** \( \{\dot{x}(t)\}_t \): The UAV flies from the initial location \( x_1 \) to the final location \( x_F \) at the maximum speed \( V_{\text{max}} \), with duration \( T = (x_F - x_1)/V_{\text{max}} \). We have \( x(t) = x_1 + V_{\text{max}}t, \quad \forall t \in (0, T] \).
- **Speed-free trajectory** \( \{\dot{x}(t)\}_t \): In this trajectory with duration \( T = T - \dot{t} \), the UAV can arbitrarily adjust its location over time without any constraints.

**Lemma 3.2:** For any given speed-constrained trajectory \( \{x(t)\}_t \) with initial and final locations \( x_1 \) and \( x_F \), we can always construct a maximum-speed trajectory \( \{\dot{x}(t)\}_t \) with duration \( \dot{T} = (x_F - x_1)/V_{\text{max}} \), and a speed-free trajectory \( \{\dot{x}(t)\}_t \) with duration \( \dot{T} = T - \dot{t} \), such that the combination of \( \{\dot{x}(t)\} \) and \( \{x(t)\} \) can achieve the same rate region achieved by \( \{x(t)\} \), i.e., \( C(x(t)) = C(\{\dot{x}(t)\}, \{x(t)\}) \), where \( C(\{\dot{x}(t)\}, \{x(t)\}) \) denotes the rate region achieved by the combination of \( \{\dot{x}(t)\} \) and \( \{x(t)\} \), as given in (11) at the top of the next page.

**Proof:** See Appendix A.

Based on Lemma 3.2, it is evident that the optimization of the speed-constrained trajectory \( \{x(t)\} \) in (P1.1) is equivalent to optimizing the maximum-speed trajectory \( \{\dot{x}(t)\} \) and the speed-free trajectory \( \{\dot{x}(t)\} \). Notice that as \( x_1 \) and \( x_F \) are fixed, the maximum-speed trajectory \( \{\dot{x}(t)\} \) is given. Therefore, we only need to optimize the speed-free trajectory \( \{\dot{x}(t)\} \).

By substituting \( C(\{x(t)\}) = C(\{\dot{x}(t)\}, \{x(t)\}) \), the speed-constrained trajectory optimization problem (P1.1) is equivalently reformulated as the following speed-free trajectory optimization problem:

(P1.2): \[
\begin{align*}
\max_{\{x(t)\}_t} & \quad R \\
\text{s.t.} & \quad r_k \geq \alpha_k R, \quad \forall k \in K \quad \text{(12)} \\
& \quad r \in C(\{\dot{x}(t)\}, \{\dot{x}(t)\}) \quad \text{(13)} \\
& \quad x_1 \leq \dot{x}(t) \leq x_F, \quad \forall t \in (0, \dot{T}] \quad \text{(14)}
\end{align*}
\]

Let \( r^* = [r_1^*, ..., r_K^*] \), \( \dot{x}^*(t) \), and \( R^* \) denote the optimal solution to problem (P1.2). Then, we have \( r^* \) and \( R^* \) as the optimal solution to (P1.1) as well. Furthermore, we can construct the optimal speed-constrained trajectory \( \{x^*(t)\} \) to (P1.1) by combining \( \{\dot{x}^*(t)\} \) together with the maximum-speed trajectory \( \{\dot{x}(t)\} \), as explained in Appendix A. Therefore, we only need to focus on solving (P1.2) next.

B. Optimal Solution to Reformulated Problem (P1.2)

Though problem (P1.2) is still non-convex, it can be shown to satisfy the so-called time-sharing condition in [22]. Therefore, the strong duality holds between (P1.2) and its Lagrange dual problem. As a result, we can optimally solve (P1.2) by applying the Lagrange duality method [23].

Let \( \lambda_k \geq 0 \) denote the dual variable associated with the \( k \)-th rate-profile constraint in (12), \( k \in K \). Then the partial Lagrangian of problem (P1.2) is

\[
\mathcal{L}(\{\lambda_k\}, \{\dot{x}(t)\}, r, R) = (1 - \sum_{k \in K} \lambda_k \alpha_k) R + \sum_{k \in K} \lambda_k r_k.
\]

(15)

Accordingly, the Lagrange dual function of (P1.2) is

\[
f(\{\lambda_k\}) = \max_{r, \{\dot{x}(t)\}_T} \mathcal{L}(\{\lambda_k\}, \{\dot{x}(t)\}, r, R) \quad \text{(16)}
\]

s.t. \((13)\) and \((14)\).

**Lemma 3.3:** In order for the dual function \( f(\{\lambda_k\}) \) to be upper-bounded from above (i.e., \( f(\{\lambda_k\}) < \infty \)), it must hold that \( \sum_{k \in K} \lambda_k \alpha_k = 1 \).

**Proof:** Suppose that \( \sum_{k \in K} \lambda_k \alpha_k > 1 \) or \( \sum_{k \in K} \lambda_k \alpha_k < 1 \).

Then by setting \( R \to -\infty \) or \( R \to \infty \), we have \( f(\{\lambda_k\}) \to \infty \). Therefore, this lemma is proved.

According to Lemma 3.3, the dual problem of (P1.2) is given by

\[
\begin{align*}
\text{(D1.2):} & \quad \min_{\{\lambda_k\}_K} f(\{\lambda_k\}) \\
\text{s.t.:} & \quad \sum_{k \in K} \lambda_k \alpha_k = 1.
\end{align*}
\]

(17)

As the strong duality holds, we can solve problem (P1.2) by equivalently solving the dual problem (D1.2). In the following, we first evaluate \( f(\{\lambda_k\}) \) in (16) under any given \( \{\lambda_k\} \), and then solve problem (D1.2) to find the optimal \( \{\lambda_k\} \), denoted by \( \{\lambda^*_k\} \).

1) **Evaluating \( f(\{\lambda_k\}) \) by Solving Problem (16):** First, we obtain the dual function \( f(\{\lambda_k\}) \) under given \( \{\lambda_k\} \) by solving problem (16). As \( \sum_{k \in K} \lambda_k \alpha_k = 1 \), the optimal solution of \( R^* \) to problem (16) can be chosen as any real value. We have \( R^* = 0 \) here for obtaining \( f(\{\lambda_k\}) \) only. Therefore, problem (16) is reduced as

\[
\max_{r, \{\dot{x}(t)\}_T} \sum_{k \in K} \lambda_k r_k
\]

s.t. \((13)\) and \((14)\).
\( \mathcal{C}(\{t(t)\}, \{x(t)\}) = \left\{ r \in \mathbb{R}^{|K|} \left| \sum_{k \in K} r_k \leq \frac{1}{T} \left( \int_0^T \log_2 \left( 1 + \frac{\sum_{k \in K} Ph_k(x(t))}{\sigma^2} \right) dt + \int_0^T \log_2 \left( 1 + \frac{\sum_{k \in K} Ph_k(x(t))}{\sigma^2} \right) dt \right) \right\}, \forall \mathcal{K} \subseteq K \right\}. \)

To solve problem (18), we have the following lemma from [21].

**Lemma 3.4:** For any given \( \{\lambda_k\} \), the optimal solution to problem (18) is obtained by a vertex \( \tilde{r}_\pi \triangleq \left[ \tilde{r}_{\pi(1)}, \ldots, \tilde{r}_{\pi(|K|)} \right] \) of the polymatroid \( \mathcal{C}(\{\tilde{x}(t)\}, \{\tilde{x}(t)\}) \), where \( \tilde{r}_{\pi(k)} \) is given as

\[ \tilde{r}_{\pi(k)} = \frac{1}{T} \int_0^T \log_2 \left( \frac{\sigma^2 + \sum_{i=1}^{k} Ph(i) (\tilde{x}(t))}{\sigma^2 + \sum_{i=1}^{k-1} Ph(i) (\tilde{x}(t))} \right) dt \]

\[ + \frac{1}{T} \int_0^T \log_2 \left( \frac{\sigma^2 + \sum_{i=1}^{k-1} Ph(i) (\tilde{x}(t))}{\sigma^2 + \sum_{i=1}^{k-1} Ph(i) (\tilde{x}(t))} \right) dt, \quad (19) \]

with the permutation \( \pi = [\pi(1), \ldots, \pi(|K|)] \) determined such that \( \lambda_{\pi(1)} \geq \cdots \geq \lambda_{\pi(|K|)} \geq 0 \).

Based on Lemma 3.4 and substituting (19), problem (16) is reformulated as problem (20) at the top of the next page, where \( \lambda_{\pi(K+1)} \triangleq 0 \) is defined for notational convenience. Notice that by dropping the constant terms, problem (20) can be decomposed into a number of subproblems in (21), each corresponding to optimizing \( \tilde{x}(t) \) for time instant \( t \in [0, T] \).

\[ \text{max} \quad \psi(\tilde{x}(t)) \]

\[ \text{subject to} \quad \sum_{k \in K} \lambda_{\pi(k)} - \lambda_{\pi(k+1)} \leq \log_2 \left( 1 + \frac{\sum_{i=1}^{k} Ph(i) (\tilde{x}(t))}{\sigma^2} \right), \quad (21) \]

It is worth noting that each subproblem in (21) is identical for different time instant \( t \in [0, T] \). As a result, we can adopt a 1D exhaustive search over the region \([x_1, x_T]\) to find the optimal \( \tilde{x} \), denoted by \( \tilde{x}^* \), which maximizes \( \psi(\tilde{x}) \) subject to \( x_1 \leq \tilde{x} \leq x_T \). Accordingly, the optimal solution to problem (20) is given by

\[ \tilde{x}^*(t) = \tilde{x}^*, \quad \forall t \in [0, T]. \]

Note that the optimal solution of \( \tilde{x}^* \) is generally non-unique, and we can arbitrarily choose any one of them for obtaining the dual function \( f(\{\lambda_k\}) \). Accordingly, we can obtain \( r^* = [r_1^*, \ldots, r_K^*] \) by using Lemma 3.4 and (19). Therefore, the dual function \( f(\{\lambda_k\}) \) is finally obtained.

2) **Finding Optimal \( \{\lambda_k^*\} \) to Solve (D1.2):** Next, with \( f(\{\lambda_k\}) \) obtained, we search over \( \{\lambda_k\} \) to minimize \( f(\{\lambda_k\}) \) for solving (D1.2). Since \( f(\{\lambda_k\}) \) is always convex but in general non-differentiable, we can use subgradient-based methods, such as the ellipsoid method [23], to obtain the optimal \( \{\lambda_k^*\} \). Note that the subgradient of the objective function \( f(\{\lambda_k\}) \) is \( s_0(\{\lambda_k\}) = [r_1^*, \ldots, r_K^*] \), while the equality constraint (17) is equivalent to two inequality constraints \( 1 - \sum_{k \in K} \lambda_k \alpha_k \leq 0 \) and

\[ -1 + \sum_{k \in K} \lambda_k \alpha_k \leq 0, \]

with subgradients being \( s_1(\{\lambda_k\}) = -\alpha \) and \( s_2(\{\lambda_k\}) = \alpha \), respectively.

3) **Constructing Optimal Primal Solution to (P1.2):** Under the optimal dual solution \( \{\lambda_k^*\} \) to (D1.2), the corresponding optimal solution of \( \tilde{x}^*(t) \), \( r^* \), and \( R^* \) to problem (16) may not be unique in general. Therefore, we need an additional step to reconstruct the optimal primal solution to (P1.2). In particular, suppose that under \( \{\lambda_k^*\} \), problem (21) has \( \Gamma \geq 1 \) optimal solutions, denoted by \( \{\tilde{x}_\gamma\}_{\gamma=1}^\Gamma \), with \( \tilde{x}_1 \leq \cdots \leq \tilde{x}_\Gamma \). In this case, to obtain the optimal primal solution to (P1.2), we need to time-share the \( \Gamma \) solutions by allowing the UAV to hover at each of the \( \Gamma \) corresponding locations for a certain duration that needs to be optimized.

On the other hand, it is also worth emphasizing that if there exist some \( \lambda_k^* \)'s that are equal to each other, then the decoding order at the UAV receiver and the corresponding average rate tuples at users are also non-unique. In this case, the UAV needs to time-share among different decoding orders as well [18]. Let \( J_1, \ldots, J_M \subseteq K \) denote \( M \) disjoint subsets such that \( \lambda_{\gamma}^* \) are identical, \( j \in J_m \), for any \( 1 \leq m \leq M \) and \( |J_m| \geq 2 \). Define set \( \mathcal{I} \triangleq \{ 1, \ldots, \prod_{m=1}^M |J_m| \} \). As a result, under the optimal dual solution \( \{\lambda_k^*\} \), problem (16) admits a number of \( |\mathcal{I}| \) optimal decoding orders, denoted by \( \pi^{(1)}, \ldots, \pi^{(|\mathcal{I}|)} \).

By combining the \( \Gamma \) hovering locations and the \( |\mathcal{I}| \) decoding orders, we have \( \Gamma \cdot |\mathcal{I}| \) associated average rate tuples, denoted by \( r^{(i)}(\tilde{x}_\gamma)'s \), \( \forall i \in \{1, \ldots, |\mathcal{I}|\}, \gamma \in \{1, \ldots, \Gamma\} \), with \( r^{(i)}(\tilde{x}_\gamma) = r_1^{(i)}(\tilde{x}_\gamma), \ldots, r_K^{(i)}(\tilde{x}_\gamma) \), which can be obtained as

\[ r_{\pi^{(i)}(\gamma)}^{(i)}(\tilde{x}_\gamma) = \frac{1}{T} \int_0^T \log_2 \left( \frac{\sigma^2 + \sum_{j=1}^{k} Ph(i) (\tilde{x}(t))}{\sigma^2 + \sum_{j=1}^{k} Ph(i) (\tilde{x}(t))} \right) dt \]

\[ + \frac{T}{T} \log_2 \left( \frac{\sigma^2 + \sum_{j=1}^{k} Ph(i) (\tilde{x}(t))}{\sigma^2 + \sum_{j=1}^{k} Ph(i) (\tilde{x}(t))} \right), \quad (23) \]

Let \( \tau^{(i)}_\gamma \) denote the normalized time-sharing factor associated with the \( \gamma \)-th hovering location and the \( i \)-th decoding order, such that the UAV uses this strategy for a \( \tau^{(i)}_\gamma \) portion of durations, where \( \sum_{\gamma=1}^\Gamma \sum_{i=1}^{|\mathcal{I}|} \tau^{(i)}_\gamma = 1 \). Accordingly, finding the optimal time-sharing factors can be formulated as the following linear program (LP), which can be solved efficiently via standard convex optimization techniques such as CVX [24].

\[ \text{(P1.3):} \quad \max_{\{r^{(i)}_\gamma \geq 0\}, R} R \]

\[ \text{s.t.} \quad \sum_{\gamma=1}^\Gamma \sum_{i=1}^{|\mathcal{I}|} \tau^{(i)}_\gamma r^{(i)}_\gamma (\tilde{x}_\gamma) \geq \alpha_k R, \quad \forall k \in K \]

\[ \sum_{\gamma=1}^\Gamma \sum_{i=1}^{|\mathcal{I}|} \tau^{(i)}_\gamma = 1 \]

Then \( R^* \) and \( \{\tau^{(i)}_\gamma\} \) denote the optimal solution to problem (P1.3). Then the UAV needs to hover at each location \( \tilde{x}_\gamma \)
for duration $\sum_{i=1}^{\Gamma} r_{\gamma}^*(t) \cdot T$. Accordingly, we partition the whole hovering period $T$ into $\Gamma$ sub-periods, denoted by $\Gamma_1, ..., \Gamma_\Gamma$, where $\Gamma_\gamma = (\sum_{\gamma'=0}^{\gamma-1} \sum_{i=1}^{\Gamma} r_{\gamma'}^*(t) \cdot T, \sum_{\gamma'=\gamma}^{\Gamma-1} \sum_{i=1}^{\Gamma} r_{\gamma'}^*(t) \cdot T, \forall \gamma \in \{1, ..., \Gamma\}$. In this case, the optimal value (or the sum-rate capacity of the UAV-enabled MAC) is given by $R^*$, and the optimal trajectory solution to the primal problem (P1.2) is given as

$$\hat{x}^*(t) = \hat{x}_{\gamma}^*, \forall t \in \Gamma_\gamma, \gamma \in \{1, ..., \Gamma\}. \quad (24)$$

Furthermore, in order to achieve the optimal communication rate $r^*$, the $i$-th decoding order $\pi(i)$ needs to be employed for a $\sum_{\gamma=1}^{\Gamma} r_{\pi(i)}^*(t)$ portion of the whole duration $T$ in total. Therefore, problem (P1.2) is finally obtained.

Remark 3.1: The optimal solution to problem (P1.2) reveals that to maximize the capacity region of the UAV-enabled MAC, the UAV needs to hover above a finite number of ground locations with optimized durations, in order to balance the rate tradeoff among different users distributed on the ground. We name such a trajectory solution as the multi-location-hovering. Besides, the UAV also needs to properly time share among different decoding orders for these users.

C. Optimal Solution to Problem (P1.1)

Now, it remains to solve problem (P1.1). First, it is evident that the optimal solution of $R^*$ and $r^*$ to (P1.2) is still optimal for (P1.1), for which the time-sharing among different decoding orders is still needed. Next, we obtain the optimal trajectory solution $\{x^*(t)\}$ to (P1.1) based on Lemma 3.2 and by combining the above multi-location-hovering solution $\{x^*(t)\}$ to problem (P1.2) and the maximum-speed trajectory $\{\hat{x}(t)\}$. We have the following proposition, for which the proof is omitted for brevity.

Proposition 3.1: The optimal trajectory solution $\{x^*(t)\}$ to (P1.1) has the following SHF structure. In particular, the UAV unidirectionally visits the initial location $x_1$, the $\Gamma$ hovering locations $\hat{x}_1, \cdots, \hat{x}_\Gamma$, and the final location $x_F$. In this trajectory, the UAV successively hovers above these locations $\hat{x}_1, \cdots, \hat{x}_\Gamma$, with durations $\sum_{\gamma=1}^{\Gamma} \Gamma_{\gamma}^*(t) \cdot T, \cdots, \sum_{\gamma=1}^{\Gamma} \Gamma_{\gamma}^*(t) \cdot T$, respectively, and flies among them at the maximum speed $V_{\text{max}}$.

Finally, by using the optimal solution to problem (P1.1) in Proposition 3.1, together with a 2D exhaustive search of $x_1$ and $x_F$, the optimal solution to the original problem (P1) is obtained. Based on Proposition 3.1, it is evident that the optimal trajectory solution to (P1) also has the interesting SHF structure.

IV. NUMERICAL RESULTS

In this section, we present numerical results to validate the performance of our proposed UAV trajectory design, in comparison to two benchmark schemes as follows.

- **Heuristic SHF trajectory**: The UAV hovers exactly above each of the $K$ ground users with optimized durations, and then flies among them at the maximum speed $V_{\text{max}}$. Notice that this trajectory is only implementable when $T \geq \frac{w_K}{V_{\text{max}}}$ in order to visit all the users.

- **Static hovering**: During the whole communication period, the UAV hovers at one single optimized location.

In the simulation, we set the UAV’s flying altitude as $H = 250$ m and the maximum speed as $V_{\text{max}} = 20$ m/s. The noise power is set as $\sigma^2 = -100$ dBm and the reference channel power gain is $\gamma_0 = -30$ dB. Furthermore, the transmit power at each user is set as $P = 30$ dBm.

![Fig. 2. Capacity region of a UAV-enabled two-user MAC with $K = 2$.](image-url)
Fig. 4 shows the common average rate of the $K$ users under different values of $T$. For comparison, we also provide the upper bound when $T \to \infty$, in which the UAV only needs to optimize its hovering locations over time. It is observed that as $T$ increases, the common average rate achieved by the proposed optimal SHF trajectory design increases considerably and approaches the performance upper bound when $T \to \infty$. It is also observed that our proposed optimal SHF trajectory design significantly outperforms the two benchmark schemes.

V. CONCLUSION

This paper characterized the fundamental capacity region of the UAV-enabled MAC in a linear topology scenario when the users are deployed in a straight line and the UAV flies at a fixed altitude. By using the rate-profile technique, we formulated the capacity-region characterization as a non-convex optimization problem consisting of an infinite of location variables over continuous time, which is very difficult to be solved optimally. We presented an efficient algorithm to obtain its globally optimal solution, by transforming the challenging speed-constrained UAV trajectory design into an equivalent speed-free trajectory optimization problem that is solvable via the Lagrange duality method. Surprisingly, we showed that the optimal UAV trajectory solution has an interesting SHF structure, i.e., the UAV unidirectionally visits a number of optimized hovering points and hovers there with certain durations, and the UAV flies among these hovering points at the maximum speed. Numerical results validated that the capacity achieved by the proposed optimal SHF trajectory design significantly outperforms the achievable rate by other benchmark schemes. It is expected that this paper can provide insights on the fundamental performance limits of UAV-enabled wireless communications, and motivate new UAV trajectory design approaches for efficient communications. How to extend the design principles to more complicated 2D or 3D trajectory optimization problems under different system setups is an interesting but challenging direction to be investigated in future.

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of this sub-period at the speed
period $T$ that by combining the two constructed trajectories
\{x(t)\} named as a speed-free trajectory. Furthermore, it is evident
generally does not satisfy any speed constraints and thus is
and
\[ \{x(t)\} \]

In the following, we construct the maximum-speed trajectory
periods each with equal duration $n$. Here, $N$ is chosen to be sufficiently large such that
the UAV flies at a constant speed $v_n \leq V_{\text{max}}$ over each sub-period $n$, i.e.,
$x(t) = x((n-1)\delta) + v_n (t - (n-1)\delta)$, $\forall t \in T_n$.
In the following, we construct the maximum-speed trajectory
\{\bar{x}(t)\} and the speed-free trajectory \{\hat{x}(t)\} by considering
each of the $N$ sub-periods, respectively.

Without loss of generality, consider one particular sub-period $T_n$, $n \in N$. First, we construct the maximum-speed sub-trajectory, in which the UAV flies from the initial location
$x((n-1)\delta)$ to the final location $x(n\delta)$ of this sub-period
at the maximum speed, with the required duration being
$\bar{\delta}_n = \frac{\delta_n}{v_{\text{max}}}$. Accordingly, we define the sub-period as $\mathcal{T}_n = \left(\sum_{i=1}^{\bar{n}-1} \delta_i, \sum_{i=1}^{\bar{n}} \delta_i\right]$, and the corresponding sub-trajectory is
$\bar{x}(t) = x((n-1)\delta) + V_{\text{max}} (t - (n-1)\delta)$, $\forall t \in \mathcal{T}_n$.
Next, we construct the other sub-trajectory, in which the UAV flies
from the initial location $x((n-1)\delta)$ to the final location $x(n\delta)$
of this sub-period at the speed $\hat{v}_n = \frac{\delta_n}{\bar{\delta}_n}$, where $\delta_n = \delta - \bar{\delta}_n$
denotes the required duration. Accordingly, we define the sub-period as $\mathcal{T}_n = \left(\sum_{i=1}^{\bar{n}-1} \delta_i, \sum_{i=1}^{\bar{n}} \delta_i\right]$, and the corresponding sub-trajectory is $\hat{x}(t) = x((n-1)\delta) + \hat{v}_n (t - \sum_{i=1}^{\bar{n}-1} \hat{\delta}_i)$, $\forall t \in \mathcal{T}_n$.
Notice that in the special case with $v_n = 0$, we have $\delta_n = 0$ for the maximum-speed sub-trajectory; while in the other special case with $v_n = V_{\text{max}}$, we have $\delta_n = 0$.

Now, by combining the sub-trajectories $\{\bar{x}(t)\}$ and $\{\hat{x}(t)\}$
over the $N$ sub-periods $\mathcal{T}_n$’s and $\mathcal{T}_n$’s, we obtain two trajectories
$\{\bar{x}(t)\}$ and $\{\hat{x}(t)\}$ with total durations $\bar{T} = (x_F - x_1)/T$
and $\hat{T} = T - \bar{T}$, respectively. It is clear that $\{\bar{x}(t)\}$ corresponds
to a maximum-speed trajectory from $x_1$ to $x_F$, while $\{\hat{x}(t)\}$
generally does not satisfy any speed constraints and thus is
named as a speed-free trajectory. Furthermore, it is evident
that by combining the two constructed trajectories $\{\bar{x}(t)\}$ and
$\{\hat{x}(t)\}$ together, the UAV visits the same locations (with the
same duration at each location) as in the original trajectory
$\{x(t)\}$. Therefore, it follows that
\[
\int_{0}^{T} \log_2 \left( 1 + \sum_{k \in \mathcal{K}} \frac{P h_k(\bar{x}(t))}{\sigma^2} \right) dt
= \int_{0}^{\bar{T}} \log_2 \left( 1 + \sum_{k \in \mathcal{K}} \frac{P h_k(\bar{x}(t))}{\sigma^2} \right) dt
+ \int_{0}^{\hat{T}} \log_2 \left( 1 + \sum_{k \in \mathcal{K}} \frac{P h_k(\hat{x}(t))}{\sigma^2} \right) dt, \forall \mathcal{K} \subseteq \mathcal{K}.
\]